

An algebraic construction of quantum flows with unbounded generators

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Abstract

It is shown how to construct $*$ -homomorphic quantum stochastic Feller cocycles for certain unbounded generators, and so obtain dilations of various strongly continuous quantum dynamical semigroups on C^* algebras. The construction is possible provided the generator satisfies an invariance property for some dense subalgebra \mathcal{A}_0 of the C^* algebra \mathcal{A} and obeys the necessary structure relations; the iterates of the generator, when applied to a generating set for \mathcal{A}_0 , must satisfy a growth condition. Furthermore, either the subalgebra \mathcal{A}_0 is generated by isometries and \mathcal{A} is universal, or \mathcal{A}_0 contains its square roots. These conditions are verified in three cases: the symmetric quantum exclusion processes, as introduced by Rebolledo, and flows on the non-commutative torus and the universal rotation algebra.

Key words: quantum dynamical semigroup; quantum Markov semigroup; CPC semigroup; strongly continuous semigroup; semigroup dilation; Feller cocycle; higher-order Itô product formula; quantum exclusion process; quantum Markov chain

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1 Introduction

In this paper we develop new techniques for solving the Evans–Hudson quantum stochastic differential equation (QSDE)

$$dj_t = \tilde{j}_t \circ \phi d\Lambda_t, \quad (1.1)$$

where \tilde{j}_t is an ampliation of the map j_t , which acts on some C^* algebra \mathcal{A} . We obtain $*$ -homomorphic solutions to the QSDE and establish that the process $(j_t)_{t \geq 0}$ is a quantum stochastic Feller cocycle. That is, it gives a Feynman–Kac perturbation of the free evolution of the surroundings, here modelled by the shift semigroup $(\sigma_t)_{t \geq 0}$ associated to the algebra of all bounded operators on \mathcal{F} , the symmetric Fock space over $L^2(\mathbb{R}_+; \mathbf{k})$; it satisfies the evolution equation

$$j_{s+t} = (j_s \bar{\otimes} \iota_{\mathcal{B}(\mathcal{F}_{[s, \infty)})}) \circ \sigma_s \circ j_t \quad (1.2)$$

for all $s, t \geq 0$, where \mathbf{k} is the multiplicity space of noise. The process j can thus also be viewed as the Markov process associated to a quantum dynamical semigroup $(T_t)_{t \geq 0}$ on \mathcal{A} . This is a

semigroup of completely positive maps on \mathcal{A} which j dilates, in the sense that $T_t = \mathbb{E} \circ j_t$ for all $t \geq 0$, where \mathbb{E} is the vacuum conditional expectation onto the subsystem \mathcal{A} obtained by averaging out the effects of the noise.

The use of quantum stochastic calculus to produce such dilations has now been studied for nearly thirty years. Most results, by Hudson and Parthasarathy, Fagnola, Mohari, Sinha *et cetera*, are obtained in the case that $\mathcal{A} = \mathcal{B}(\mathfrak{h})$ by first solving an operator-valued QSDE, the Hudson–Parthasarathy equation, to obtain a unitary process U , and defining j through conjugation by U ; see [8] and references therein. The corresponding theory from the Heisenberg viewpoint, solving the Evans–Hudson equation (1.1), has mainly been developed under the standing assumption that the generator ϕ is completely bounded, which is necessary if the corresponding semigroup T is norm continuous [16]. When one deviates from this assumption, which is analytically convenient but very restrictive, there are few results. The earliest general method is due to Fagnola and Sinha [9], with later results by Goswami, Sahu and Sinha for a particular model [12] and a more general method developed by Goswami and Sinha in [23]. Another approach based on semigroup methods has yet to yield existence results for the Evans–Hudson equation: see [1] and [20].

Our method here has an attractive simplicity, imposing minimal conditions on the generator ϕ . It must be a $*$ -linear map

$$\phi : \mathcal{A}_0 \rightarrow \mathcal{A}_0 \otimes \mathcal{B}(\mathbb{C} \oplus \mathfrak{k}),$$

where \mathcal{A}_0 is a dense $*$ -subalgebra of the unital C^* algebra $\mathcal{A} \subseteq \mathcal{B}(\mathfrak{h})$ which contains $1 = 1_{\mathfrak{h}}$. This incorporates an assumption that, if ϕ is viewed as a matrix of maps, its components leave \mathcal{A}_0 invariant, a hypothesis also used in [9]. Furthermore, ϕ must be such that $\phi(1) = 0$ and the *first-order Itô product formula* holds:

$$\phi(xy) = \phi(x)(y \otimes 1_{\widehat{\mathfrak{k}}}) + (x \otimes 1_{\widehat{\mathfrak{k}}})\phi(y) + \phi(x)\Delta\phi(y) \quad \text{for all } x, y \in \mathcal{A}_0, \quad (1.3)$$

where $\widehat{\mathfrak{k}} := \mathbb{C} \oplus \mathfrak{k}$ and $\Delta \in \mathcal{A}_0 \otimes \mathcal{B}(\widehat{\mathfrak{k}})$ is the orthogonal projection onto $\mathfrak{h} \bar{\otimes} \mathfrak{k}$. Both these conditions are known to be necessary if ϕ is to generate a family of unital $*$ -homomorphisms. Finally, a growth bound must be established for the iterates of ϕ applied to elements taken from a suitable subset of \mathcal{A}_0 .

Our approach is an elementary one, relying on familiar techniques such as representing the solution to the Evans–Hudson QSDE as a sum of quantum Wiener integrals. An essential tool is the higher-order Itô product formula, presented in Section 2. This formula was first stated, for finite-dimensional noise, in [6], was proved for that case in [13] and reached its definitive form in [19]. In that last paper it was shown that (1.3) is but the first of a sequence of identities that must be satisfied in order to show that the solution j of the QSDE is weakly multiplicative. However, there are many situations in which the validity of (1.3) implies that the other identities hold [19, Corollary 4.2], and this is the case for ϕ as above. Moreover, one of our main observations, Corollary 2.11, is that this product formula also allows us to utilise pointwise bounds at individual elements to provide a bound at their product; in this way, one need only establish these bounds for a $*$ -generating set of \mathcal{A}_0 ; this is a major simplification on the approach in [9]. Also, by using the coordinate-free approach to quantum stochastic analysis

given in [15], we can take \mathbf{k} to be any Hilbert space, rather than insisting that \mathbf{k} be finite dimensional, as in [9].

The growth bounds obtained Section 2 are employed in Section 3 to produce a family of weakly multiplicative $*$ -linear maps from the algebra \mathcal{A}_0 into the space of linear operators in $\mathbf{h} \otimes \mathcal{F}$ with domains which include $\mathbf{h} \otimes \mathcal{E}$. It is shown that these maps extend to give a $*$ -homomorphic cocycle in two distinct situations. Theorem 3.9, which includes the case of AF algebras, exploits a square-root trick that is well known in the literature; Theorem 3.12, which applies to universal C^* algebras such as the non-commutative torus or the Cuntz algebras, is believed to be novel. Uniqueness of the solution is proved, and it is also shown that j is a cocycle, *i.e.*, it satisfies (1.2). At this point we see another novel feature of our work in contrast to previous results, all of which start with a particular quantum dynamical semigroup T . In these other papers the generator τ of T is then augmented to produce ϕ , and the QSDE solved to give a dilation of ϕ . For example, in [9] it is assumed that T is an analytic semigroup and that the composition of τ with the other components of ϕ is well behaved in a certain sense; in [23] it is assumed that T is covariant with respect to some group action on \mathcal{A} . For us, the starting point is the map ϕ , which yields the cocycle j , and hence, by compression, a quantum dynamical semigroup T that, *a fortiori*, is dilated by j . Thus we do not have to check that τ is a semigroup generator with good properties at the outset, thereby rendering our method easier to apply.

Theorem 3.9 is employed in Section 4 to produce a dilation of the symmetric quantum exclusion semigroup. This object was introduced by Rebolledo [21] and has generated much interest: see [11] and [10]. This example, in common with previous work on processes arising in quantum interacting particle systems (*e.g.*, [12]) requires \mathbf{k} to be infinite dimensional.

In Section 5 we use Theorem 3.12 to obtain flows on some universal C^* algebras, namely the non-commutative torus and the universal rotation algebra [2]. Quantum flows on these algebras have previously been considered by Goswami, Sahu and Sinha [12] and by Hudson and Robinson [14], respectively.

1.1 Conventions and notation

The quantity $:=$ is to be read as ‘is defined to be’ or similarly. The quantity $\mathbb{1}_P$ equals 1 if the proposition P is true and 0 if P is false, where 1 and 0 are the appropriate multiplicative and additive identities. The set of natural numbers is denoted by $\mathbb{N} := \{1, 2, 3, \dots\}$; the set of non-negative integers is denoted by $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$. The linear span of the set S in the vector space V is denoted by $\text{lin } S$; all vector spaces have complex scalar field and inner products are linear on the right. The *algebraic* tensor product is denoted by \otimes ; the Hilbert-space tensor product is denoted by $\bar{\otimes}$, as is the ultraweak tensor product. The domain of the linear operator T is denoted by $\text{dom}(T)$. The identity transformation on the vector space V is denoted by 1_V . If P is an orthogonal projection on the inner-product space V then the complement $P^\perp := 1_V - P$, the projection onto the orthogonal complement of the range of P . The Banach space of bounded operators from the Banach space X_1 to the Banach space X_2 is denoted by $\mathcal{B}(X_1; X_2)$, or by $\mathcal{B}(X_1)$ if X_1 and X_2 are equal. The identity automorphism on

the algebra \mathcal{A} is denoted by $\iota_{\mathcal{A}}$. If a and b are elements in an algebra \mathcal{A} then $[a, b] := ab - ba$ and $\{a, b\} := ab + ba$ denote their commutator and anti-commutator, respectively. If \mathcal{A}_0 is a $*$ -algebra, H_1 and H_2 are Hilbert spaces and $\alpha : \mathcal{A}_0 \rightarrow \mathcal{B}(H_1; H_2)$ is a linear map then the *adjoint* map $\alpha^\dagger : \mathcal{A}_0 \rightarrow \mathcal{B}(H_2; H_1)$ is such that $\alpha^\dagger(a) := \alpha(a^*)^*$ for all $a \in \mathcal{A}_0$.

2 A higher-order product formula

Notation 2.1. The Dirac bra-ket notation will be useful: for any Hilbert space H and vectors $\xi, \chi \in H$, let

$$\begin{aligned} |H\rangle &:= \mathcal{B}(\mathbb{C}; H), & |\xi\rangle &: \mathbb{C} \rightarrow H; \lambda \mapsto \lambda\xi & (ket) \\ \text{and} \quad \langle H| &:= \mathcal{B}(H; \mathbb{C}), & \langle\chi| &: H \rightarrow \mathbb{C}; \eta \mapsto \langle\chi, \eta\rangle & (bra). \end{aligned}$$

In particular, we have the linear map $|\xi\rangle\langle\chi| \in \mathcal{B}(H)$ such that $|\xi\rangle\langle\chi|\eta = \langle\chi, \eta\rangle\xi$ for all $\eta \in H$.

Let $\mathcal{A} \subseteq \mathcal{B}(\mathfrak{h})$ be a unital C^* algebra with identity $1 = 1_{\mathfrak{h}}$, whose elements act as bounded operators on the *initial space* \mathfrak{h} , a Hilbert space. Let $\mathcal{A}_0 \subseteq \mathcal{A}$ be a norm-dense $*$ -subalgebra of \mathcal{A} which contains 1.

Let the *extended multiplicity space* $\widehat{\mathbf{k}} := \mathbb{C} \oplus \mathbf{k}$, where the *multiplicity space* \mathbf{k} is a Hilbert space, and distinguish the unit vector $\omega := (1, 0)$. For brevity, let $\mathcal{B} := \mathcal{B}(\widehat{\mathbf{k}})$.

Let $\Delta := 1 \otimes P_{\mathbf{k}} \in \mathcal{A}_0 \otimes \mathcal{B}$, where $P_{\mathbf{k}} := |\omega\rangle\langle\omega|^\perp \in \mathcal{B}$ is the orthogonal projection onto $\mathbf{k} \subset \widehat{\mathbf{k}}$.

Lemma 2.2. *The map $\phi : \mathcal{A}_0 \rightarrow \mathcal{A}_0 \otimes \mathcal{B}$ is $*$ -linear, such that $\phi(1) = 0$ and such that*

$$\phi(xy) = \phi(x)(y \otimes 1_{\mathbf{k}}) + (x \otimes 1_{\mathbf{k}})\phi(y) + \phi(x)\Delta\phi(y) \quad \text{for all } x, y \in \mathcal{A}_0 \quad (2.1)$$

if and only if

$$\phi(x) = \begin{bmatrix} \tau(x) & \delta^\dagger(x) \\ \delta(x) & \pi(x) - x \otimes 1_{\mathbf{k}} \end{bmatrix} \quad \text{for all } x \in \mathcal{A}_0, \quad (2.2)$$

where $\pi : \mathcal{A}_0 \rightarrow \mathcal{A}_0 \otimes \mathcal{B}(\mathbf{k})$ is a unital $$ -homomorphism, $\delta : \mathcal{A}_0 \rightarrow \mathcal{A}_0 \otimes |\mathbf{k}\rangle$ is a π -derivation, i.e., a linear map such that*

$$\delta(xy) = \delta(x)y + \pi(x)\delta(y) \quad \text{for all } x, y \in \mathcal{A}_0,$$

and $\tau : \mathcal{A}_0 \rightarrow \mathcal{A}_0$ is a $$ -linear map such that*

$$\tau(xy) - \tau(x)y - x\tau(y) = \delta^\dagger(x)\delta(y) \quad \text{for all } x, y \in \mathcal{A}_0. \quad (2.3)$$

Proof. This is a straightforward exercise in elementary algebra. \square

Definition 2.3. A $*$ -linear map $\phi : \mathcal{A}_0 \rightarrow \mathcal{A}_0 \otimes \mathcal{B}$ such that $\phi(1) = 0$ and such that (2.1) holds is a *flow generator*.

Lemma 2.4. Let $\mathcal{A}_0 = \mathcal{A}$, let $\pi : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{B}(\mathbf{k})$ be a unital $*$ -homomorphism, let $z \in \mathcal{A} \otimes |\mathbf{k}\rangle$ and let $h \in \mathcal{A}$ be self adjoint. Define

$$\delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{B}(\mathbb{C}; \mathbf{k}); \quad x \mapsto zx - \pi(x)z$$

and

$$\tau : \mathcal{A} \rightarrow \mathcal{A}; \quad x \mapsto i[h, x] - \frac{1}{2}\{z^*z, x\} + z^*\pi(x)z.$$

Then the map $\phi : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{B}$ defined in terms of π , δ and τ through (2.2) is a flow generator.

Proof. This is another straightforward exercise. \square

Remark 2.5. Modulo important considerations regarding tensor products and the ranges of δ and τ , the above form for ϕ is, essentially, the only one possible [17, Lemma 6.4]. The quantum exclusion process in Section 4 has a generator of the same form but with unbounded z and h .

Definition 2.6. Given a flow generator $\phi : \mathcal{A}_0 \rightarrow \mathcal{A}_0 \otimes \mathcal{B}$, the *quantum random walk* $(\phi_n)_{n \in \mathbb{Z}_+}$ is a family of $*$ -linear maps

$$\phi_n : \mathcal{A}_0 \rightarrow \mathcal{A}_0 \otimes \mathcal{B}^{\otimes n}$$

defined by setting

$$\phi_0 := \iota_{\mathcal{A}_0} \quad \text{and} \quad \phi_{n+1} := (\phi_n \otimes \iota_{\mathcal{B}}) \circ \phi \quad \text{for all } n \in \mathbb{Z}_+.$$

The following identity is useful: if $\xi_1, \chi_1, \dots, \xi_n, \chi_n \in \widehat{\mathbf{k}}$ and $x \in \mathcal{A}_0$ then

$$(1_{\mathbf{h}} \otimes \langle \xi_1 | \otimes \dots \otimes \langle \xi_n |) \phi_n(x) (1_{\mathbf{h}} \otimes |\chi_1\rangle \otimes \dots \otimes |\chi_n\rangle) = \phi_{\chi_1}^{\xi_1} \circ \dots \circ \phi_{\chi_n}^{\xi_n}(x), \quad (2.4)$$

where

$$\phi_{\chi}^{\xi} : \mathcal{A}_0 \rightarrow \mathcal{A}_0; \quad x \mapsto (1_{\mathbf{h}} \otimes \langle \xi |) \phi(x) (1_{\mathbf{h}} \otimes |\chi\rangle)$$

is a linear map for each choice of $\xi, \chi \in \widehat{\mathbf{k}}$.

Remark 2.7. The article [19], results from which will be employed below, uses a different convention to that adopted in Definition 2.6: the components of the product $\mathcal{B}^{\otimes n}$ appear in the reverse order to how they do above.

Notation 2.8. Let $\alpha \subseteq \{1, \dots, n\}$, with elements arranged in increasing order, and denote its cardinality by $|\alpha|$. The unital $*$ -homomorphism

$$\mathcal{A}_0 \otimes \mathcal{B}^{\otimes |\alpha|} \rightarrow \mathcal{A}_0 \otimes \mathcal{B}^{\otimes n}; \quad T \mapsto T(n, \alpha)$$

is defined by linear extension of the map

$$A \otimes B_1 \otimes \dots \otimes B_{|\alpha|} \mapsto A \otimes C_1 \otimes \dots \otimes C_n,$$

where

$$C_i := \begin{cases} B_j & \text{if } i \text{ is the } j\text{th element of } \alpha, \\ 1_{\widehat{\mathbf{k}}} & \text{if } i \text{ is not an element of } \alpha. \end{cases}$$

For example, if $\alpha = \{1, 3, 4\}$ then

$$(A \otimes B_1 \otimes B_2 \otimes B_3)(5, \alpha) = A \otimes B_1 \otimes 1_{\widehat{\mathbf{k}}} \otimes B_2 \otimes B_3 \otimes 1_{\widehat{\mathbf{k}}}.$$

Given a flow generator $\phi : \mathcal{A}_0 \rightarrow \mathcal{A}_0 \otimes \mathcal{B}$, for all $n \in \mathbb{Z}_+$ and $\alpha \subseteq \{1, \dots, n\}$, let

$$\phi_{|\alpha|}(x; n, \alpha) := (\phi_{|\alpha|}(x))(n, \alpha) \quad \text{for all } x \in \mathcal{A}_0$$

and let

$$\Delta(n, \alpha) := (1_{\mathbf{h}} \otimes P_{\mathbf{k}}^{\otimes |\alpha|})(n, \alpha),$$

so that, in the latter, $P_{\mathbf{k}}$ acts on the components of $\widehat{\mathbf{k}}^{\otimes n}$ which have indices in α and $1_{\widehat{\mathbf{k}}}$ acts on the others.

Theorem 2.9. *Let $(\phi_n)_{n \in \mathbb{Z}_+}$ be the quantum random walk given by the flow generator ϕ . For all $n \in \mathbb{Z}_+$ and $x, y \in \mathcal{A}_0$,*

$$\phi_n(xy) = \sum_{\alpha \cup \beta = \{1, \dots, n\}} \phi_{|\alpha|}(x; n, \alpha) \Delta(n, \alpha \cap \beta) \phi_{|\beta|}(y; n, \beta), \quad (2.5)$$

where the summation is taken over all sets α and β whose union is $\{1, \dots, n\}$.

Proof. This may be established inductively: see [19, Proof of Theorem 4.1]. \square

Definition 2.10. The set $S \subseteq \mathcal{A}_0$ is **-generating* for \mathcal{A}_0 if \mathcal{A}_0 is the smallest unital *-algebra which contains S .

Corollary 2.11. *For a flow generator $\phi : \mathcal{A}_0 \rightarrow \mathcal{A}_0 \otimes \mathcal{B}$, let*

$$\mathcal{A}_\phi := \{x \in \mathcal{A}_0 : \text{there exist } C_x, M_x > 0 \text{ such that } \|\phi_n(x)\| \leq C_x M_x^n \text{ for all } n \in \mathbb{Z}_+\}. \quad (2.6)$$

*Then \mathcal{A}_ϕ is a *-subalgebra of \mathcal{A}_0 , which is equal to \mathcal{A}_0 if \mathcal{A}_ϕ contains a *-generating set for \mathcal{A}_0 .*

Proof. It suffices to demonstrate that \mathcal{A}_ϕ is closed under products. To see this, let $x, y \in \mathcal{A}_\phi$ and suppose C_x, M_x and C_y, M_y are as in (2.6). Then (2.5) implies that

$$\begin{aligned} \|\phi_n(xy)\| &\leq \sum_{\alpha \cup \beta = \{1, \dots, n\}} \|\phi_{|\alpha|}(x)\| \|\phi_{|\beta|}(y)\| \\ &\leq C_x C_y \sum_{k=0}^n \binom{n}{k} M_x^k \sum_{l=0}^k \binom{k}{l} M_y^{n-k+l} \quad (k = |\alpha|, l = |\alpha \cap \beta|) \\ &= C_x C_y \sum_{k=0}^n \binom{n}{k} M_x^k M_y^{n-k} (1 + M_y)^k \\ &= C_x C_y (M_x + M_x M_y + M_y)^n \end{aligned}$$

for all $n \in \mathbb{Z}_+$, as required. \square

Lemma 2.12. *If the flow generator ϕ is as defined in Lemma 2.4 then $\mathcal{A}_\phi = \mathcal{A}_0$.*

Proof. This follows immediately, since ϕ is completely bounded and $\|\phi_n\| \leq \|\phi_n\|_{\text{cb}} \leq \|\phi\|_{\text{cb}}^n$ for all $n \in \mathbb{Z}_+$. \square

Lemma 2.13. *Let $\phi : \mathcal{A}_0 \rightarrow \mathcal{A}_0 \otimes \mathcal{B}$ be a flow generator. For all $\xi, \chi \in \widehat{\mathbf{k}}$ we have $\phi_\chi^\xi(\mathcal{A}_\phi) \subseteq \mathcal{A}_\phi$, and the series*

$$\exp(z\phi_\chi^\xi) := \sum_{n=0}^{\infty} \frac{z^n (\phi_\chi^\xi)^n}{n!} \quad (2.7)$$

is strongly absolutely convergent on \mathcal{A}_ϕ for all $z \in \mathbb{C}$.

Proof. Suppose $\|\phi_n(x)\| \leq C_x M_x^n$ for all $n \in \mathbb{Z}_+$. It follows from (2.4) that

$$(1_{\mathfrak{h} \otimes \widehat{\mathbf{k}}^{\otimes n}} \otimes \langle \xi |) \phi_{n+1}(x) (1_{\mathfrak{h} \otimes \widehat{\mathbf{k}}^{\otimes n}} \otimes |\chi\rangle) = \phi_n(\phi_\chi^\xi(x)), \quad (2.8)$$

so

$$\|\phi_n(\phi_\chi^\xi(x))\| \leq \|\xi\| C_x M_x^{n+1} \|\chi\| = (\|\xi\| \|\chi\| C_x M_x) M_x^n$$

and $\phi_\chi^\xi(x) \in \mathcal{A}_\phi$. Moreover (2.4) also gives that

$$\|(\phi_{\chi_1}^{\xi_1} \circ \cdots \circ \phi_{\chi_n}^{\xi_n})(x)\| \leq \|\xi_1\| \cdots \|\xi_n\| \|\chi_1\| \cdots \|\chi_n\| C_x M_x^n, \quad (2.9)$$

hence the series (2.7) converges as claimed. \square

3 Quantum flows

Notation 3.1. Let \mathcal{F} denote Boson Fock space over $L^2(\mathbb{R}_+; \mathbf{k})$, the Hilbert space of \mathbf{k} -valued, square-integrable functions on the half line, and let

$$\mathcal{E} := \text{lin}\{\varepsilon(f) : f \in L^2(\mathbb{R}_+; \mathbf{k})\}$$

denote the linear span of the total set of exponential vectors in \mathcal{F} . As is customary, elementary tensors in $\mathfrak{h} \otimes \mathcal{F}$ are written without a tensor-product sign: in other words, $u\varepsilon(f) := u \otimes \varepsilon(f)$ for all $u \in \mathfrak{h}$ and $f \in L^2(\mathbb{R}_+; \mathbf{k})$, *et cetera*.

If $f \in L^2(\mathbb{R}_+; \mathbf{k})$ and $t \geq 0$ then $\widehat{f}(t) := \widehat{f(t)}$, where $\widehat{\xi} := \omega + \xi \in \widehat{\mathbf{k}}$ for all $\xi \in \mathbf{k}$.

Given $f \in L^2(\mathbb{R}_+; \mathbf{k})$ and an interval $I \subseteq \mathbb{R}_+$, let $f_I \in L^2(\mathbb{R}_+; \mathbf{k})$ be defined to equal f on I and 0 elsewhere, with $f_t := f_{[0,t)}$ and $f_{[t} := f_{[t,\infty)}$ for all $t \geq 0$.

Definition 3.2. A family of linear operators $(T_t)_{t \geq 0}$ in $\mathfrak{h} \otimes \mathcal{F}$ with domains including $\mathfrak{h} \otimes \mathcal{E}$ is *adapted* if

$$\langle u\varepsilon(f), T_t v\varepsilon(g) \rangle = \langle u\varepsilon(f_t), T_t v\varepsilon(g_t) \rangle \langle \varepsilon(f_{[t}), \varepsilon(g_{[t}) \rangle$$

for all $u, v \in \mathfrak{h}$, $f, g \in L^2(\mathbb{R}_+; \mathbf{k})$ and $t \geq 0$.

Theorem 3.3. For all $n \in \mathbb{N}$ and $T \in \mathcal{B}(\mathfrak{h} \bar{\otimes} \widehat{\mathfrak{k}}^{\otimes n})$ there exists a family $\Lambda^n(T) = (\Lambda_t^n(T))_{t \geq 0}$ of linear operators in $\mathfrak{h} \bar{\otimes} \mathcal{F}$, with domains including $\mathfrak{h} \otimes \mathcal{E}$, that is adapted and such that

$$\langle u\varepsilon(f), \Lambda_t^n(T)v\varepsilon(g) \rangle = \int_{D_n(t)} \langle u \otimes \widehat{f}^{\otimes n}(\mathbf{t}), Tv \otimes \widehat{g}^{\otimes n}(\mathbf{t}) \rangle d\mathbf{t} \langle \varepsilon(f), \varepsilon(g) \rangle \quad (3.1)$$

for all $u, v \in \mathfrak{h}$, $f, g \in L^2(\mathbb{R}_+; \mathfrak{k})$ and $t \geq 0$. Here the simplex

$$D_n(t) := \{\mathbf{t} := (t_1, \dots, t_n) \in [0, t]^n : t_1 < \dots < t_n\}$$

and

$$\widehat{f}^{\otimes n}(\mathbf{t}) := \widehat{f}(t_1) \otimes \dots \otimes \widehat{f}(t_n), \quad \text{et cetera.}$$

We extend this definition to include $n = 0$ by setting $\Lambda_t^0(T) := T \otimes 1_{\mathcal{F}}$ for all $t \geq 0$.

If $f \in L^2(\mathbb{R}_+; \mathfrak{k})$ then

$$\|\Lambda_t^n(T)u\varepsilon(f)\| \leq \frac{K_{f,t}^n}{\sqrt{n!}} \|T\| \|u\varepsilon(f)\| \quad \text{for all } t \geq 0 \text{ and } u \in \mathfrak{h}, \quad (3.2)$$

where $K_{f,t} := \sqrt{(2 + 4\|f\|^2)(t + \|f\|^2)}$, and the map

$$\mathbb{R}_+ \rightarrow \mathcal{B}(\mathfrak{h}; \mathfrak{h} \bar{\otimes} \mathcal{F}); \quad t \mapsto \Lambda_t^n(T)(1_{\mathfrak{h}} \otimes |\varepsilon(f)\rangle)$$

is norm continuous.

Proof. This is an extension of Proposition 3.18 of [15], from which we borrow the notation; as for Remark 2.7, the ordering of the components in tensor products is different but this is no more than a convention. For each $f \in L^2(\mathbb{R}_+; \mathfrak{k})$ define $C_f \geq 0$ so that

$$C_f^2 = (\|f\| + \sqrt{1 + \|f\|^2})^2 \leq 2 + 4\|f\|^2,$$

and note that, by inequality (3.21) of [15],

$$\begin{aligned} \|\Lambda_t^n(T)u\varepsilon(f)\|^2 &\leq (C_{f,t})^{2n} \int_{D_n(t)} \|Tu \otimes \widehat{f}^{\otimes n}(\mathbf{t})\|^2 d\mathbf{t} \|\varepsilon(f)\|^2 \\ &\leq \frac{K_{f,t}^{2n}}{n!} \|T\|^2 \|u\varepsilon(f)\|^2. \end{aligned}$$

To show continuity, let \widetilde{T} denote T considered as an operator on $(\mathfrak{h} \bar{\otimes} \widehat{\mathfrak{k}}) \bar{\otimes} \widehat{\mathfrak{k}}^{\otimes(n-1)}$, where the right-most copy of $\widehat{\mathfrak{k}}$ in the n -fold tensor product has moved next to the initial space \mathfrak{h} . Then

$$\Lambda_t^n(T) - \Lambda_s^n(T) = \Lambda_t(1_{(s,t]}(\cdot) \Lambda^{n-1}(\widetilde{T})),$$

and so, using Theorem 3.13 of [15],

$$\begin{aligned} \|(\Lambda_t^n(T) - \Lambda_s^n(T))u\varepsilon(f)\|^2 &\leq 2(t + C_f^2) \int_s^t \|\Lambda_r^{n-1}(\widetilde{T})(u \otimes \widehat{f}(r))\varepsilon(f)\|^2 dr \\ &\leq 2(t + C_f^2) \left(\int_s^t \|\widehat{f}(r)\|^2 dr \right) \frac{K_{f,t}^{2n-2}}{(n-1)!} \|T\|^2 \|u\varepsilon(f)\|^2. \quad \square \end{aligned}$$

The family $\Lambda^n(T)$ is the n -fold quantum Wiener integral of T .

Remark 3.4. It may be shown [19, Proof of Theorem 2.2] that

$$\text{dom}(\Lambda_t^l(S)^*) \supseteq \Lambda_t^m(T)(\mathfrak{h} \otimes \mathcal{E})$$

for all $l, m \in \mathbb{Z}_+$, $S \in \mathcal{B}(\mathfrak{h} \bar{\otimes} \widehat{\mathfrak{k}}^{\bar{\otimes} l})$, $T \in \mathcal{B}(\mathfrak{h} \bar{\otimes} \widehat{\mathfrak{k}}^{\bar{\otimes} m})$ and $t \geq 0$.

Theorem 3.5. Let $\phi : \mathcal{A}_0 \rightarrow \mathcal{A}_0 \otimes \mathcal{B}$ be a flow generator. If $x \in \mathcal{A}_\phi$ then the series

$$j_t(x) := \sum_{n=0}^{\infty} \Lambda_t^n(\phi_n(x)) \quad (3.3)$$

is strongly absolutely convergent on $\mathfrak{h} \otimes \mathcal{E}$ for all $t \geq 0$, uniformly so on compact subsets of \mathbb{R}_+ . The map

$$\mathbb{R}_+ \rightarrow \mathcal{B}(\mathfrak{h}; \mathfrak{h} \bar{\otimes} \mathcal{F}); \quad t \mapsto j_t(x)(1_{\mathfrak{h}} \otimes |\varepsilon(f)\rangle)$$

is norm continuous for all $f \in L^2(\mathbb{R}_+; \mathfrak{k})$, the family $(j_t(x))_{t \geq 0}$ is adapted and

$$\langle u\varepsilon(f), j_t(x)v\varepsilon(g) \rangle = \langle u\varepsilon(f), (xv)\varepsilon(g) \rangle + \int_0^t \langle u\varepsilon(f), j_s(\phi_{\widehat{g}(s)}^{\widehat{f}(s)}(x))v\varepsilon(g) \rangle ds \quad (3.4)$$

for all $u, v \in \mathfrak{h}$, $f, g \in L^2(\mathbb{R}_+; \mathfrak{k})$ and $t \geq 0$. Furthermore,

$$(1_{\mathfrak{h}} \otimes \langle \varepsilon(f) |) j_t(x) (1_{\mathfrak{h}} \otimes |\varepsilon(g)\rangle) \in \mathcal{A} \quad (3.5)$$

for all $x \in \mathcal{A}_\phi$, $f, g \in L^2(\mathbb{R}_+; \mathfrak{k})$ and $t \geq 0$.

Proof. The first two claims are a consequence of the estimate (3.2), the definition of \mathcal{A}_ϕ and the continuity result from Theorem 3.3; adaptedness is inherited from the adaptedness of the quantum Wiener integrals. Lemma 2.13 implies that the integrand on the right-hand side of (3.4) is well defined and, by (2.8),

$$\begin{aligned} & \langle u\varepsilon(f), \Lambda_s^n(\phi_n(\phi_{\widehat{g}(s)}^{\widehat{f}(s)}(x)))v\varepsilon(g) \rangle \\ &= \int_{D_n(s)} \langle u \otimes \widehat{f}^{\otimes n}(\mathbf{t}), \phi_n(\phi_{\widehat{g}(s)}^{\widehat{f}(s)}(x))v \otimes \widehat{g}^{\otimes n}(\mathbf{t}) \rangle d\mathbf{t} \langle \varepsilon(f), \varepsilon(g) \rangle \\ &= \int_{D_n(s)} \langle u \otimes \widehat{f}^{\otimes n}(\mathbf{t}) \otimes \widehat{f}(s), \phi_{n+1}(x)v \otimes \widehat{g}^{\otimes n}(\mathbf{t}) \otimes \widehat{g}(s) \rangle d\mathbf{t} \langle \varepsilon(f), \varepsilon(g) \rangle; \end{aligned}$$

integrating with respect to s then taking the sum of these terms gives (3.4). For the final claim, note that for any $f, g \in L^2(\mathbb{R}_+; \mathfrak{k})$, the \mathcal{A}_0 -valued map

$$D_n(t) \ni \mathbf{t} \mapsto \phi_{\widehat{g}(t_1)}^{\widehat{f}(t_1)} \circ \cdots \circ \phi_{\widehat{g}(t_n)}^{\widehat{f}(t_n)}(x) = (1_{\mathfrak{h}} \otimes \langle \widehat{f}^{\otimes n}(\mathbf{t}) |) \phi_n(x) (1_{\mathfrak{h}} \otimes |\widehat{g}^{\otimes n}(\mathbf{t})\rangle)$$

is Bochner integrable, hence

$$(1_{\mathfrak{h}} \otimes \langle \varepsilon(f) |) \Lambda_t^n(\phi_n(x)) (1_{\mathfrak{h}} \otimes |\varepsilon(g)\rangle) = e^{\langle f, g \rangle} \int_{D_n(t)} (\phi_{\widehat{g}(t_1)}^{\widehat{f}(t_1)} \circ \cdots \circ \phi_{\widehat{g}(t_n)}^{\widehat{f}(t_n)})(x) dt \in \mathcal{A}. \quad (3.6)$$

By (2.9), we may sum (3.6) over all $n \in \mathbb{Z}_+$, with the resulting series being norm convergent, and so the final claim follows. \square

Remark 3.6. For all $t \geq 0$, let j_t be as in Theorem 3.5. Since ϕ is linear with $\phi(1) = 0$, we have from (3.4) that $j_t(1) = 1_{\mathfrak{h} \otimes \mathcal{E}}$ and $j_t(x + y) = j_t(x) + j_t(y)$, as operators on $\mathfrak{h} \otimes \mathcal{E}$, for all $x, y \in \mathcal{A}_\phi$.

Lemma 3.7. Let $\phi : \mathcal{A}_0 \rightarrow \mathcal{A}_0 \otimes \mathcal{B}$ be a flow generator and let j_t be as in Theorem 3.5 for all $t \geq 0$. If $x, y \in \mathcal{A}_\phi$ then $x^*y \in \mathcal{A}_\phi$, with

$$\langle j_t(x)u\varepsilon(f), j_t(y)v\varepsilon(g) \rangle = \langle u\varepsilon(f), j_t(x^*y)v\varepsilon(g) \rangle \quad (3.7)$$

for all $u, v \in \mathfrak{h}$ and $f, g \in L^2(\mathbb{R}_+; \mathfrak{k})$. In particular, if $x \in \mathcal{A}_\phi$ then $j_t(x)^* \supseteq j_t(x^*)$.

Proof. As \mathcal{A}_ϕ is a $*$ -algebra, so $x^*y \in \mathcal{A}_\phi$. Let $N \in \mathbb{Z}_+$ and note that, by [19, Theorem 2.2],

$$\sum_{l,m=0}^N \Lambda_t^l(\phi_l(x))^* \Lambda_t^m(\phi_m(y)) = \sum_{n=0}^{2N} \Lambda_t^n(\phi_{n,N}](x^*y)) \quad \text{on } \mathfrak{h} \otimes \mathcal{E}, \quad (3.8)$$

where

$$\phi_{n,N}](x^*y) := \sum_{\substack{\alpha \cup \beta = \{1, \dots, n\} \\ |\alpha|, |\beta| \leq N}} \phi_{|\alpha|}(x^*; n, \alpha) \Delta(n, \alpha \cap \beta) \phi_{|\beta|}(y; n, \beta).$$

Working as in the proof of Corollary 2.11 yields the inequality

$$\|\phi_{n,N}](x^*y)\| \leq C_{x^*} C_y (M_{x^*} + M_{x^*} M_y + M_y)^n,$$

and so, by (3.2),

$$|\langle u\varepsilon(f), \Lambda_t^n(\phi_{n,N}](x^*y))v\varepsilon(g) \rangle| \leq \frac{K_{g,t}^n (M_{x^*} + M_{x^*} M_y + M_y)^n}{\sqrt{n!}} C_{x^*} C_y \|u\varepsilon(f)\| \|v\varepsilon(g)\|. \quad (3.9)$$

As $\phi_{n,N}] = \phi_n$ if $n \in \{0, 1, \dots, N\}$, it follows that

$$\begin{aligned} \langle j_t(x)u\varepsilon(f), j_t(y)v\varepsilon(g) \rangle &= \lim_{N \rightarrow \infty} \sum_{l,m=0}^N \langle u\varepsilon(f), \Lambda_t^l(\phi_l(x))^* \Lambda_t^m(\phi_m(y))v\varepsilon(g) \rangle \\ &= \lim_{N \rightarrow \infty} \sum_{n=0}^N \langle u\varepsilon(f), \Lambda_t^n(\phi_n(x^*y))v\varepsilon(g) \rangle \\ &\quad + \lim_{N \rightarrow \infty} \sum_{n=N+1}^{2N} \langle u\varepsilon(f), \Lambda_t^n(\phi_{n,N}](x^*y))v\varepsilon(g) \rangle \\ &= \langle u\varepsilon(f), j_t(x^*y)v\varepsilon(g) \rangle, \end{aligned}$$

since the final limit is zero by (3.9). \square

Lemma 3.8. *If \mathcal{A}_ϕ is dense in \mathcal{A} then there is at most one family of $*$ -homomorphisms $(\bar{j}_t)_{t \geq 0}$ from \mathcal{A} to $\mathcal{B}(\mathfrak{h} \bar{\otimes} \mathcal{F})$ such that $\bar{j}_t(x)|_{\mathfrak{h} \otimes \mathcal{E}} = j_t(x)$ for all $x \in \mathcal{A}_\phi$ and $t \geq 0$, where $j_t(x)$ is as defined in Theorem 3.5.*

Proof. Suppose that $j^{(1)}$ and $j^{(2)}$ are two families of $*$ -homomorphisms from \mathcal{A} to $\mathcal{B}(\mathfrak{h} \otimes \mathcal{F})$ such that $j_t^{(i)}(x)|_{\mathfrak{h} \otimes \mathcal{E}} = j_t(x)$ for $i = 1, 2$. Set $k_t := j_t^{(1)} - j_t^{(2)}$ and note that, from (3.4), we have that

$$\langle u\varepsilon(f), k_t(x)v\varepsilon(g) \rangle = \int_0^t \langle u\varepsilon(f), k_s(\phi_{\hat{g}(s)}^{\hat{f}(s)}(x))v\varepsilon(g) \rangle ds$$

for all $u, v \in \mathfrak{h}$, $f, g \in L^2(\mathbb{R}_+; \mathfrak{k})$ and $x \in \mathcal{A}_\phi$. Iterating the above, and using the fact that $\|k_t\| \leq 2$ for all $t \geq 0$, we obtain the inequality

$$|\langle u\varepsilon(f), k_t(x)v\varepsilon(g) \rangle| \leq 2 \int_{D_n(t)} \|\phi_{\hat{g}(t_1)}^{\hat{f}(t_1)} \circ \dots \circ \phi_{\hat{g}(t_n)}^{\hat{f}(t_n)}(x)\| dt \|u\varepsilon(f)\| \|v\varepsilon(g)\|.$$

However (2.9) now gives that

$$|\langle u\varepsilon(f), k_t(x)v\varepsilon(g) \rangle| \leq 2C_x \frac{(M_x \|\hat{f}_t\| \|\hat{g}_t\|)^n}{n!} \|u\varepsilon(f)\| \|v\varepsilon(g)\|,$$

and the result follows by letting $n \rightarrow \infty$. □

Theorem 3.9. *Let $\phi : \mathcal{A}_0 \rightarrow \mathcal{A}_0 \otimes \mathcal{B}$ be a flow generator and suppose \mathcal{A}_0 contains its square roots: for all non-negative $x \in \mathcal{A}_0$, the square root $x^{1/2}$ lies in \mathcal{A}_0 . If $\mathcal{A}_\phi = \mathcal{A}_0$ then, for all $t \geq 0$, there exists a unital $*$ -homomorphism*

$$\bar{j}_t : \mathcal{A} \rightarrow \mathcal{B}(\mathfrak{h} \bar{\otimes} \mathcal{F})$$

such that $\bar{j}_t(x) = j_t(x)$ on $\mathfrak{h} \otimes \mathcal{E}$ for all $x \in \mathcal{A}_0$, where $j_t(x)$ is as defined in Theorem 3.5.

Proof. Let $x \in \mathcal{A}_0$ and suppose first that $x \geq 0$. If $y := (\|x\|1 - x)^{1/2}$, which lies in \mathcal{A}_0 by assumption, then Lemma 3.7 and Remark 3.6 imply that

$$0 \leq \|j_t(y)\theta\|^2 = \langle \theta, j_t(y^2)\theta \rangle = \|x\| \|\theta\|^2 - \langle \theta, j_t(x)\theta \rangle \quad \text{for all } \theta \in \mathfrak{h} \otimes \mathcal{E}.$$

If x is now an arbitrary element of \mathcal{A}_0 , it follows that

$$\|j_t(x)\theta\|^2 = \langle \theta, j_t(x^*x)\theta \rangle \leq \|x^*x\| \|\theta\|^2 = \|x\|^2 \|\theta\|^2.$$

Thus $j_t(x)$ extends to $\bar{j}_t(x) \in \mathcal{B}(\mathfrak{h} \bar{\otimes} \mathcal{F})$, which has norm at most $\|x\|$, and the map

$$\mathcal{A}_0 \rightarrow \mathcal{B}(\mathfrak{h} \bar{\otimes} \mathcal{F}); \quad x \mapsto \bar{j}_t(x)$$

is a $*$ -linear contraction, which itself extends to a $*$ -linear contraction

$$\bar{j}_t : \mathcal{A} \rightarrow \mathcal{B}(\mathfrak{h} \bar{\otimes} \mathcal{F}).$$

Furthermore, if $x, y \in \mathcal{A}_0$ and $\theta, \zeta \in \mathfrak{h} \otimes \mathcal{E}$ then, by Lemma 3.7,

$$\langle \theta, \bar{j}_t(x) \bar{j}_t(y) \zeta \rangle = \langle \bar{j}_t(x^*) \theta, \bar{j}_t(y) \zeta \rangle = \langle j_t(x^*) \theta, j_t(y) \zeta \rangle = \langle \theta, j_t(xy) \zeta \rangle = \langle \theta, \bar{j}_t(xy) \zeta \rangle,$$

so \bar{j}_t is multiplicative on \mathcal{A}_0 . An approximation argument now gives that \bar{j}_t is multiplicative on the whole of \mathcal{A} . \square

Remark 3.10. If \mathcal{A} is an AF algebra, *i.e.*, the norm closure of an increasing sequence of finite-dimensional $*$ -subalgebras, then its local algebra \mathcal{A}_0 , the union of these subalgebras, contains its square roots, since every finite-dimensional C^* algebra is closed in \mathcal{A} .

Definition 3.11. The unital C^* algebra \mathcal{A} has *generators* $\{a_i : i \in I\}$ if \mathcal{A} is the smallest unital C^* algebra which contains $\{a_i : i \in I\}$. These generators *satisfy the relations* $\{p_k : k \in K\}$ if each p_k is a complex polynomial in the non-commuting indeterminate $\langle X_i, X_i^* : i \in I \rangle$ and, for all $k \in K$, the algebra element $p_k(a_i, a_i^* : i \in I)$, obtained from p_k by replacing X_i by a_i and X_i^* by a_i^* for all $i \in I$, is equal to 0.

Suppose \mathcal{A} has generators $\{a_i : i \in I\}$ which satisfy the relations $\{p_k : k \in K\}$. Then \mathcal{A} is *generated by isometries* if $\{X_i^* X_i - 1 : i \in I\} \subseteq \{p_k : k \in K\}$ and is *generated by unitaries* if $\{X_i^* X_i - 1, X_i X_i^* - 1 : i \in I\} \subseteq \{p_k : k \in K\}$. The algebra \mathcal{A} is *universal* if, given any unital C^* algebra \mathcal{B} containing a set of elements $\{b_i : i \in I\}$ which satisfies the relations $\{p_k : k \in K\}$, *i.e.*, $p_k(b_i, b_i^* : i \in I) = 0$ for all $k \in K$, there exists a unique $*$ -homomorphism $\pi : \mathcal{A} \rightarrow \mathcal{B}$ such that $\pi(a_i) = b_i$ for all $i \in I$.

Theorem 3.12. Let \mathcal{A} be the universal C^* algebra generated by isometries $\{s_i : i \in I\}$ which satisfy the relations $\{p_k : k \in K\}$, and let \mathcal{A}_0 be the $*$ -algebra generated by $\{s_i : i \in I\}$. If $\phi : \mathcal{A}_0 \rightarrow \mathcal{A}_0 \otimes \mathcal{B}$ is a flow generator such that $\mathcal{A}_\phi = \mathcal{A}_0$ then, for all $t \geq 0$, there exists a unital $*$ -homomorphism

$$\bar{j}_t : \mathcal{A} \rightarrow \mathcal{B}(\mathfrak{h} \bar{\otimes} \mathcal{F})$$

such that $\bar{j}_t(x) = j_t(x)$ on $\mathfrak{h} \otimes \mathcal{E}$ for all $x \in \mathcal{A}_0$, where $j_t(x)$ is as defined in Theorem 3.5.

Proof. Remark 3.6 and Lemma 3.7 imply that $j_t(s_i)$ is isometric and that $j_t(s_i^*)$ is contractive for all $i \in I$. Repeated application of (3.7) then shows that $j_t(x)$ is bounded for each $x \in \mathcal{A}_0$, and that j_t extends to a unital $*$ -homomorphism from \mathcal{A}_0 to $\mathcal{B}(\mathfrak{h} \bar{\otimes} \mathcal{F})$. Furthermore, the set $\{j_t(s_i) : i \in I\}$ satisfies the relations $\{p_k : k \in K\}$ so, by the universal nature of \mathcal{A} , there exists a $*$ -homomorphism π from \mathcal{A} into $\mathcal{B}(\mathfrak{h} \bar{\otimes} \mathcal{F})$ such that $\pi(s_i) = j_t(s_i)$ for all $i \in I$ and $\bar{j}_t := \pi$ is as required. \square

Corollary 3.13. The family $(\bar{j}_t : \mathcal{A} \rightarrow \mathcal{B}(\mathfrak{h} \bar{\otimes} \mathcal{F}))_{t \geq 0}$ constructed in Theorems 3.9 and 3.12 is a strong solution of the QSDE (1.1).

Proof. Fix $x \in \mathcal{A}_\phi$ and let

$$L_t := \sigma((\bar{j}_t \otimes \iota_{\mathcal{B}})(\phi(x)))$$

for all $t \geq 0$, where $\sigma : \mathcal{B}(\mathfrak{h} \bar{\otimes} \mathcal{F} \bar{\otimes} \widehat{\mathfrak{k}}) \rightarrow \mathcal{B}(\mathfrak{h} \bar{\otimes} \widehat{\mathfrak{k}} \bar{\otimes} \mathcal{F})$ is the automorphism that swaps the last two components of simple tensors. If $f \in L^2(\mathbb{R}_+; \mathfrak{k})$ then

$$\|L_t u \otimes \widehat{f}(t) \otimes \varepsilon(f)\| \leq \|\phi(x)\| \|\widehat{f}(t)\| \|u \varepsilon(f)\|,$$

so if $t \mapsto L_t u \otimes \widehat{f}(t) \otimes \varepsilon(f)$ is strongly measurable then $t \mapsto L_t$ is quantum stochastically integrable [15, p.232] and \bar{j} satisfies the QSDE in the strong sense, since we already have from (3.4) that it is a weak solution.

Now, Theorem 3.5 implies that for each $x \in \mathcal{A}_\phi = \mathcal{A}_0$ and $\theta \in \mathfrak{h} \otimes \mathcal{E}$ the map $t \mapsto \bar{j}_t(x)\theta$ is continuous, hence so is

$$t \mapsto (\bar{j}_t \otimes \iota_{\mathcal{B}})(y \otimes T)(\theta \otimes \xi) = \bar{j}_t(y)\theta \otimes T\xi$$

for all $y \in \mathcal{A}_0$, $T \in \mathcal{B}(\widehat{\mathfrak{k}})$ and $\xi \in \widehat{\mathfrak{k}}$. As $\|L_t\| = \|\phi(x)\|$ for all $t \geq 0$, it follows that $t \mapsto L_t$ and $t \mapsto L_t^*$ are strongly continuous on $\mathfrak{h} \bar{\otimes} \widehat{\mathfrak{k}} \bar{\otimes} \mathcal{F}$. Hence $t \mapsto L_t(u \otimes \widehat{f}(t) \otimes \varepsilon(f))$ is separably valued and weakly measurable, so Pettis's theorem gives the result. \square

Remark 3.14. Property (3.5) implies that the homomorphism \bar{j}_t given by Theorems 3.9 and 3.12 takes values in the matrix space $\mathcal{A} \otimes_{\mathbb{M}} \mathcal{B}(\mathcal{F})$.

Notation 3.15. For all $t \geq 0$, $f, g \in L^2(\mathbb{R}_+; \mathfrak{k})$ and $a \in \mathcal{A}$, let

$$\bar{j}_t[f, g](a) := (1_{\mathfrak{h}} \otimes \langle \varepsilon(f_t) |) \bar{j}_t(a) (1_{\mathfrak{h}} \otimes |\varepsilon(g_t)\rangle).$$

Theorem 3.16. *The family of $*$ -homomorphisms $(\bar{j}_t)_{t \geq 0}$ given by Theorems 3.9 and 3.12 forms a Feller cocycle [18, Section 2.4] for the shift semigroup on $\mathcal{B}(\mathcal{F})$: for all $s, t \geq 0$, $f, g \in L^2(\mathbb{R}_+; \mathfrak{k})$ and $a \in \mathcal{A}$,*

- (i) $\bar{j}_0[0, 0](a) = a$,
 - (ii) $\bar{j}_t[f, g](a) \in \mathcal{A}$,
 - (iii) $t \mapsto \bar{j}_t[f, g](a)$ is norm continuous
- and (iv) $\bar{j}_{s+t}[f, g] = \bar{j}_s[f, g] \circ \bar{j}_t[f(\cdot + s), g(\cdot + s)]$.

Consequently, setting

$$T_t(a) := \bar{j}_t[0, 0](a) = (1_{\mathfrak{h}} \otimes \langle \varepsilon(0) |) \bar{j}_t(a) (1_{\mathfrak{h}} \otimes |\varepsilon(0)\rangle) \quad \text{for all } a \in \mathcal{A}$$

gives a strongly continuous semigroup $T = (T_t)_{t \geq 0}$ of completely positive contractions on \mathcal{A} such that $T_t(x) = \exp(t\phi_\omega^\omega)(x)$ for all $x \in \mathcal{A}_0$ and $t \geq 0$. In particular, $T_t(1) = 1$ for all $t \geq 0$ and \mathcal{A}_0 is a core for the generator of T .

Proof. Properties (i) and (ii) are immediate consequences of (3.4) and (3.5) respectively. For (iii), note that if $x \in \mathcal{A}_0$ and $f, g \in L^2(\mathbb{R}_+; \mathfrak{k})$ then Theorem 3.5 implies that

$$t \mapsto \bar{j}_t[f, g](x) = (1_{\mathfrak{h}} \otimes \langle \varepsilon(f) |) \bar{j}_t(x) (1_{\mathfrak{h}} \otimes |\varepsilon(g)\rangle) \exp\left(-\int_t^\infty \langle f(s), g(s) \rangle ds\right)$$

is norm continuous; the general case follows by approximation.

In order to establish (iv), fix $s \geq 0$ and continuous functions $f, g \in L^2(\mathbb{R}_+; \mathbf{k})$, and let

$$J_t := \bar{j}_s[f, g] \circ \bar{j}_t[f(\cdot + s), g(\cdot + s)] \quad \text{for all } t \geq 0.$$

We will show that $J_t = \bar{j}_{s+t}[f, g]$.

First note that for any $x \in \mathcal{A}_0$ and $t > 0$, the map

$$F : [0, t] \rightarrow \mathcal{A}; \quad r \mapsto \bar{j}_r[f(\cdot + s), g(\cdot + s)](\phi_{\hat{g}(r+s)}^{\hat{f}(r+s)}(x)) \langle \varepsilon(f_{[s+r, s+t]}), \varepsilon(g_{[s+r, s+t]}) \rangle$$

is continuous, hence Bochner integrable, and so

$$x \langle \varepsilon(f_{[s, s+t]}), \varepsilon(g_{[s, s+t]}) \rangle + \int_0^t F(r) \, dr \in \mathcal{A}.$$

By the adaptedness of $\bar{j}_t(x)$ and (3.4),

$$\begin{aligned} & \langle u, \left(x \langle \varepsilon(f_{[s, s+t]}), \varepsilon(g_{[s, s+t]}) \rangle + \int_0^t F(r) \, dr \right) v \rangle \\ &= \langle u, xv \rangle \langle \varepsilon(f(\cdot + s)_t), \varepsilon(g(\cdot + s)_t) \rangle \\ & \quad + \int_0^t \langle u \varepsilon(f(\cdot + s)_r), j_r(\phi_{\hat{g}(r+s)}^{\hat{f}(r+s)}(x)) v \varepsilon(g(\cdot + s)_r) \rangle \langle \varepsilon(f(\cdot + s)_{[r, t]}), \varepsilon(g(\cdot + s)_{[r, t]}) \rangle \, dr \\ &= \langle u \varepsilon(f(\cdot + s)_t), j_t(x) v \varepsilon(g(\cdot + s)_t) \rangle \\ &= \langle u, \bar{j}_t[f(\cdot + s), g(\cdot + s)](x) v \rangle. \end{aligned}$$

Consequently,

$$\begin{aligned} \langle u, J_t(x) v \rangle &= \langle u, \bar{j}_s[f, g](x) v \rangle \langle \varepsilon(f_{[s, s+t]}), \varepsilon(g_{[s, s+t]}) \rangle \\ & \quad + \int_0^t \langle u, \bar{j}_s[f, g] \circ \bar{j}_r[f(\cdot + s), g(\cdot + s)](\phi_{\hat{g}(r+s)}^{\hat{f}(r+s)}(x)) v \rangle \langle \varepsilon(f_{[s+r, s+t]}), \varepsilon(g_{[s+r, s+t]}) \rangle \, dr \\ &= \langle u, \bar{j}_s[f, g](x) v \rangle \langle \varepsilon(f_{[s, s+t]}), \varepsilon(g_{[s, s+t]}) \rangle \\ & \quad + \int_0^t \langle u, J_r(\phi_{\hat{g}(r+s)}^{\hat{f}(r+s)}(x)) v \rangle \langle \varepsilon(f_{[s+r, s+t]}), \varepsilon(g_{[s+r, s+t]}) \rangle \, dr. \end{aligned}$$

On the other hand, by (3.4),

$$\begin{aligned} \langle u, \bar{j}_{s+t}[f, g](x) v \rangle &= \langle u, xv \rangle \langle \varepsilon(f_{s+t}), \varepsilon(g_{s+t}) \rangle + \int_0^s \langle u \varepsilon(f_{s+t}), j_r(\phi_{\hat{g}(r)}^{\hat{f}(r)}(x)) v \varepsilon(g_{s+t}) \rangle \, dr \\ & \quad + \int_s^{s+t} \langle u \varepsilon(f_{s+t}), j_r(\phi_{\hat{g}(r)}^{\hat{f}(r)}(x)) v \varepsilon(g_{s+t}) \rangle \, dr \\ &= \langle u \varepsilon(f_{s+t}), j_s(x) v \varepsilon(g_{s+t}) \rangle + \int_0^t \langle u \varepsilon(f_{s+t}), j_{q+s}(\phi_{\hat{g}(q+s)}^{\hat{f}(q+s)}(x)) v \varepsilon(g_{s+t}) \rangle \, dq \\ &= \langle u, \bar{j}_s[f, g](x) v \rangle \langle \varepsilon(f_{[s, s+t]}), \varepsilon(g_{[s, s+t]}) \rangle \\ & \quad + \int_0^t \langle u, \bar{j}_{q+s}[f, g](\phi_{\hat{g}(q+s)}^{\hat{f}(q+s)}(x)) v \rangle \langle \varepsilon(f_{[s+q, s+t]}), \varepsilon(g_{[s+q, s+t]}) \rangle \, dq. \end{aligned}$$

Now set $K_t := J_t - \bar{j}_{s+t}[f, g]$, so that

$$\langle u, K_t(x)v \rangle = \int_0^t \langle u, K_r(\phi_{\hat{g}(r+s)}^{\hat{f}(r+s)}(x))v \rangle G(r) dr,$$

where $G : r \mapsto \langle \varepsilon(f_{[s+r, s+t]}), \varepsilon(g_{[s+r, s+t]}) \rangle$ is continuous. As

$$\|K_t\| \leq 2 \exp\left(\frac{1}{2}(\|f\|^2 + \|g\|^2)\right) \quad \text{for all } t \geq 0,$$

iterating the above and estimating as in the proof of Lemma 3.8 gives that $K \equiv 0$. The density of \mathcal{A}_0 in \mathcal{A} and of continuous functions in $L^2(\mathbb{R}_+; \mathfrak{h})$ now gives (iv).

That T is a semigroup follows from this cocycle property (iv): note that

$$T_{s+t} = \bar{j}_{s+t}[0, 0] = \bar{j}_s[0, 0] \circ \bar{j}_t[0, 0] = T_s \circ T_t \quad \text{for all } s, t \geq 0.$$

Contractivity, complete positivity and strong continuity of T are immediate; the exponential identity holds because

$$\langle u, T_t(x)v \rangle = \langle u, xv \rangle + \int_0^t \langle u, T_s(\phi_\omega^\omega(x))v \rangle ds \quad (3.10)$$

for all $u, v \in \mathfrak{h}$, $t \geq 0$ and $x \in \mathcal{A}_0$, by (3.4). That \mathcal{A}_0 is a core for the generator of T follows from Lemma 2.13 and [3, Corollary 3.1.20]. \square

Remark 3.17. A $*$ -homomorphic Feller cocycle as in Theorem 3.16 is called a *quantum flow*; a strongly continuous semigroup $(T_t)_{t \geq 0}$ of completely positive contractions is known as a *quantum dynamical semigroup*, and the condition $T_t(1) = 1$ for all $t \geq 0$ means that the semigroup is *conservative*; conservative quantum dynamical semigroups are also known as *quantum Markov semigroups*. Hence Theorem 3.16 gives the existence of a quantum flow which dilates a quantum Markov semigroup on the C^* algebra \mathcal{A} .

4 The symmetric quantum exclusion process

This section was inspired by Rebolledo's treatment of the quantum exclusion process: see [21, Examples 2.4.3 and 4.1.3].

Definition 4.1. Let I be a non-empty set. The *CAR algebra* is the unital C^* algebra \mathcal{A} with generators $\{b_i : i \in I\}$, subject to the anti-commutation relations

$$\{b_i, b_j\} = 0 \quad \text{and} \quad \{b_i, b_j^*\} = \mathbb{1}_{i=j} \quad \text{for all } i, j \in I. \quad (4.1)$$

It follows from (4.1) that the b_i are nonzero partial isometries for all $i \in I$.

As is well known [4, Proposition 5.2.2], \mathcal{A} is represented faithfully and irreducibly on $\mathcal{F}_-(\ell^2(I))$, the Fermionic Fock space over $\ell^2(I)$; in other words, we may (and do) suppose that $\mathcal{A} \subseteq \mathcal{B}(\mathfrak{h})$, where $\mathfrak{h} := \mathcal{F}_-(\ell^2(I))$, and the algebra identity $1 = 1_{\mathfrak{h}}$.

Remark 4.2. The elements of I may be taken to correspond to sites at which Fermionic particles may exist, with the operators b_i and b_i^* representing the annihilation and creation, respectively, of a particle at site i .

Notation 4.3. Let \mathcal{A}_0 be the unital algebra generated by $\{b_i, b_i^* : i \in I\}$; by definition, this is a norm-dense unital $*$ -subalgebra of \mathcal{A} .

Lemma 4.4. *For each $x \in \mathcal{A}_0$ there exists a finite subset $J \subseteq I$ such that x lies in the finite-dimensional $*$ -subalgebra*

$$\mathcal{A}_J := \text{lin}\{b_{j_1}^* \cdots b_{j_q}^* b_{i_1} \cdots b_{i_p} : 0 \leq p, q \leq |J|, \{i_1, \dots, i_p\} \in J^{(p)}, \{j_1, \dots, j_q\} \in J^{(q)}\} \subseteq \mathcal{A}_0,$$

where $J^{(p)}$ denote the set of subsets of J with cardinality p et cetera. Consequently, \mathcal{A} is an AF algebra and \mathcal{A}_0 contains its square roots.

Proof. By employing the anti-commutation relations (4.1), any finite product of terms from the generating set $\{b_i, b_i^* : i \in I\}$ may be reduced to a linear combination of words of the form

$$b_{j_1}^* \cdots b_{j_q}^* b_{i_1} \cdots b_{i_p}, \quad (4.2)$$

where i_1, \dots, i_p are distinct elements of I , as are j_1, \dots, j_q , and $p, q \in \mathbb{Z}_+$, with an empty product equal to 1. As every element of \mathcal{A}_0 is a finite linear combination of such terms, the first claim follows. The second claim holds by Remark 3.10. \square

Definition 4.5. Let $\{\alpha_{i,j} : i, j \in I\} \subseteq \mathbb{C}$ be a fixed collection of *amplitudes*. We may view $(I, \{\alpha_{i,j} : i, j \in I\})$ as a weighted directed graph, where I is the set of vertices, an edge exists from i to j if $\alpha_{i,j} \neq 0$ and $\alpha_{i,j}$ is a complex weight on the edge from vertex i to vertex j , which may differ from the weight $\alpha_{j,i}$ from j to i .

For all $i \in I$, let

$$\text{supp}(i) := \{j \in I : \alpha_{i,j} \neq 0\} \quad \text{and} \quad \text{supp}^+(i) := \text{supp}(i) \cup \{i\}.$$

Thus $\text{supp}(i)$ is the set of sites with which site i interacts and $|\text{supp}(i)|$ is the valency of the vertex i . We require that the valencies are finite:

$$|\text{supp}(i)| < \infty \quad \text{for all } i \in I. \quad (4.3)$$

The transport of a particle from site i to site j with amplitude $\alpha_{i,j}$ is described by the operator

$$t_{i,j} := \alpha_{i,j} b_j^* b_i.$$

Definition 4.6. Let $\{\eta_i : i \in I\} \subseteq \mathbb{R}$ be fixed. The total energy in the system is given by

$$h := \sum_{i \in I} \eta_i b_i^* b_i,$$

where η_i gives the energy of a particle at site i . If the set $\{i \in I : \eta_i \neq 0\}$ is infinite then the proper interpretation of h involves issues of convergence; below it will only appear in a commutator with elements of \mathcal{A}_0 , which is sufficient to give a well-defined quantity.

Lemma 4.7. *Let*

$$\tau_{i,j}(x) := t_{i,j}^*[t_{i,j}, x] + [x, t_{i,j}^*]t_{i,j} = |\alpha_{i,j}|^2 (b_i^* b_j [b_j^* b_i, x] + [x, b_i^* b_j] b_j^* b_i)$$

for all $i, j \in I$ and $x \in \mathcal{A}$, and let

$$[h, x] := \sum_{i \in I} \eta_i [b_i^* b_i, x] \quad (4.4)$$

for all $x \in \mathcal{A}_0$. Setting

$$\tau(x) := i[h, x] - \frac{1}{2} \sum_{i,j \in I} \tau_{i,j}(x) \quad (4.5)$$

defines a $*$ -linear map $\tau : \mathcal{A}_0 \rightarrow \mathcal{A}_0$.

Proof. Let $x \in \mathcal{A}_0$ and note that $x \in \mathcal{A}_J$ for some finite set $J \subseteq I$, by Lemma 4.4. Furthermore,

$$[b_j^* b_i, x] = b_j^* \{b_i, x\} - \{b_j^*, x\} b_i = 0 \quad \text{whenever } i \notin J \text{ and } j \notin J,$$

so

$$[h, x] = \sum_{i \in J} \eta_i [b_i^* b_i, x] \in \mathcal{A}_J \quad \text{and} \quad \tau(x) = i[h, x] - \frac{1}{2} \sum_{i,j \in J^+} \tau_{i,j}(x) \in \mathcal{A}_{J^+},$$

where

$$J^+ := \bigcup_{k \in J} \text{supp}^+(k). \quad (4.6)$$

Hence $\tau(\mathcal{A}_J) \subseteq \mathcal{A}_{J^+}$ and, as (4.3) implies that J^+ is finite, it follows that \mathcal{A}_0 is invariant under τ . The $*$ -linearity of τ is immediately verified. \square

Lemma 4.8. *Let*

$$\delta_{i,j}(x) := [t_{i,j}, x] = \alpha_{i,j} (b_j^* b_i x - x b_j^* b_i)$$

for all $i, j \in I$ and $x \in \mathcal{A}$, and let \mathbf{k} be a Hilbert space with orthonormal basis $\{f_{i,j} : i, j \in I\}$. Setting

$$\delta(x) := \sum_{i,j \in I} \delta_{i,j}(x) \otimes |f_{i,j}\rangle \quad (4.7)$$

for all $x \in \mathcal{A}_0$ defines a linear map $\delta : \mathcal{A}_0 \rightarrow \mathcal{A}_0 \otimes |\mathbf{k}\rangle$ such that

$$\delta(xy) = \delta(x)y + (x \otimes 1_{\mathbf{k}})\delta(y) \quad (4.8)$$

$$\text{and} \quad \delta^\dagger(x)\delta(y) = \tau(xy) - \tau(x)y - x\tau(y) \quad (4.9)$$

for all $x, y \in \mathcal{A}_0$, where τ is as defined in Lemma 4.7.

Proof. The series in (4.7) contains only finitely many terms, since if $x \in \mathcal{A}_J$ then

$$\delta_{i,j}(x) = 0 \quad \text{when } \{i, j\} \not\subseteq J^+.$$

Hence δ is well defined, and (4.8) holds because each $\delta_{i,j}$ is a derivation. A short calculation shows that

$$\tau_{i,j}(xy) - \tau_{i,j}(x)y - x\tau_{i,j}(y) = -2\delta_{i,j}^\dagger(x)\delta_{i,j}(y) \quad (4.10)$$

for all $x, y \in \mathcal{A}$. Since $x \mapsto [b_i^*b_i, x]$ is a derivation for all $i \in I$, it follows from (4.10) that

$$\tau(xy) - \tau(x)y - x\tau(y) = \sum_{i,j \in I} \delta_{i,j}^\dagger(x)\delta_{i,j}(y) = \delta^\dagger(x)\delta(y) \quad \text{for all } x, y \in \mathcal{A}_0. \quad \square$$

Lemma 4.9. *The map*

$$\phi : \mathcal{A}_0 \rightarrow \mathcal{A}_0 \otimes \mathcal{B}; \quad x \mapsto \begin{bmatrix} \tau(x) & \delta^\dagger(x) \\ \delta(x) & 0 \end{bmatrix}, \quad (4.11)$$

where τ , δ and δ^\dagger are as defined in Lemmas 4.7 and 4.8, is a flow generator.

If the amplitudes satisfy the symmetry condition

$$|\alpha_{i,j}| = |\alpha_{j,i}| \quad \text{for all } i, j \in I \quad (4.12)$$

then, for all $n \in \mathbb{N}$ and $i_0 \in I$,

$$\phi_n(b_{i_0}) = \sum_{i_1 \in \text{supp}^+(i_0)} \cdots \sum_{i_n \in \text{supp}^+(i_{n-1})} b_{i_n} \otimes B_{i_{n-1}, i_n} \otimes \cdots \otimes B_{i_0, i_1}, \quad (4.13)$$

where

$$B_{i,j} := \mathbb{1}_{j=i} \lambda_i |\omega\rangle \langle \omega| + |\omega\rangle \langle \alpha_{i,j} f_{i,j}| - |\alpha_{j,i} f_{j,i}\rangle \langle \omega|$$

and

$$\lambda_i := -i\eta_i - \frac{1}{2} \sum_{j \in \text{supp}(i)} |\alpha_{j,i}|^2$$

for all $i, j \in I$.

Proof. The first claim is an immediate consequence of Lemmas 4.7, 4.8 and 2.2.

If $i, j, k \in I$ then a short calculation shows that

$$\tau_{j,k}(b_i) = \begin{cases} |\alpha_{i,i}|^2 b_i & (j = i, k = i), \\ |\alpha_{j,i}|^2 b_j^* b_j b_i & (j \neq i, k = i), \\ |\alpha_{i,k}|^2 b_k b_k^* b_i & (j = i, k \neq i), \\ 0 & (j \neq i, k \neq i). \end{cases}$$

Since

$$[h, b_i] = \sum_{j \in I} \eta_j [b_j^* b_j, b_i] = \eta_i [b_i^* b_i, b_i] = -\eta_i b_i,$$

the symmetry condition (4.12) implies that

$$\tau(b_i) = \lambda_i b_i \quad \text{for all } i \in I.$$

Furthermore, if $i, j, k \in I$ then

$$\delta_{j,k}(b_i) = \alpha_{j,k}(b_k^* b_j b_i - b_i b_k^* b_j) = -\alpha_{j,k}\{b_k^*, b_i\} b_j = -\mathbb{1}_{k=i} \alpha_{j,i} b_j$$

and

$$\delta_{j,k}^\dagger(b_i) = \overline{\alpha_{j,k}}(b_i b_j^* b_k - b_j^* b_k b_i) = \overline{\alpha_{j,k}}\{b_i, b_j^*\} b_k = \mathbb{1}_{j=i} \overline{\alpha_{i,k}} b_k;$$

thus

$$\delta(b_i) = \sum_{j,k \in I} \delta_{j,k}(b_i) \otimes |f_{j,k}\rangle = - \sum_{j \in \text{supp}(i)} \alpha_{j,i} b_j \otimes |f_{j,i}\rangle$$

and

$$\delta^\dagger(b_i) = \sum_{j,k \in I} \delta_{j,k}^\dagger(b_i) \otimes \langle f_{j,k}| = \sum_{k \in \text{supp}(i)} \overline{\alpha_{i,k}} b_k \otimes \langle f_{i,k}|.$$

Hence

$$\begin{aligned} \phi(b_i) &= \lambda_i b_i \otimes |\omega\rangle\langle\omega| - \sum_{j \in \text{supp}(i)} \alpha_{j,i} b_j \otimes |f_{j,i}\rangle\langle\omega| + \sum_{k \in \text{supp}(i)} \overline{\alpha_{i,k}} b_k \otimes |\omega\rangle\langle f_{i,k}| \\ &= \sum_{j \in \text{supp}^+(i)} b_j \otimes (\mathbb{1}_{j=i} \lambda_i |\omega\rangle\langle\omega| + |\omega\rangle\langle\alpha_{i,j} f_{i,j}| - |\alpha_{j,i} f_{j,i}\rangle\langle\omega|) \end{aligned}$$

and the identity (4.13) follows. \square

Theorem 4.10. *Let \mathcal{A} be the CAR algebra and let ϕ be defined as in Lemma 4.9. If the amplitudes $\{\alpha_{i,j}\}$ and energies $\{\eta_i\}$ are chosen so that $\mathcal{A}_\phi = \mathcal{A}_0$ then there exists an adapted family of $*$ -homomorphisms $(j_t : \mathcal{A} \rightarrow \mathcal{B}(\mathfrak{h} \bar{\otimes} \mathcal{F}))_{t \geq 0}$ that forms a Feller cocycle in the sense of Theorem 3.16 and satisfies the quantum stochastic differential equation (1.1) in the strong sense on \mathcal{A}_0 for all $t \geq 0$. Setting*

$$T_t(a) := (1_{\mathfrak{h}} \otimes \langle \varepsilon(0)|) j_t(a) (1_{\mathfrak{h}} \otimes |\varepsilon(0)\rangle) \quad \text{for all } a \in \mathcal{A} \text{ and } t \geq 0$$

gives a conservative quantum dynamical semigroup T on \mathcal{A} whose generator is the closure of

$$\tau : \mathcal{A}_0 \rightarrow \mathcal{A}_0; \quad x \mapsto i \sum_{i \in I} \eta_i [b_i^* b_i, x] - \frac{1}{2} \sum_{i,j \in I} |\alpha_{i,j}|^2 (b_i^* b_j [b_j^* b_i, x] + [x, b_i^* b_j] b_j^* b_i).$$

Proof. This is an immediate consequence of Theorem 3.9, Theorem 3.16 and Lemma 4.9. \square

Example 4.11. Suppose that the amplitudes satisfy the symmetry condition (4.12), and further that there are uniform bounds on the amplitudes, valencies and energies:

$$M := \sup_{i,j \in I} |\alpha_{i,j}| < \infty, \quad V := \sup_{i \in I} |\text{supp}(i)| < \infty \quad \text{and} \quad H := \sup_{i \in I} |\eta_i| < \infty. \quad (4.14)$$

It follows that

$$|\lambda_i| \leq |\eta_i| + \frac{1}{2}VM^2 \quad \text{and} \quad \|B_{i,j}\| \leq |\lambda_i| + 2M \leq H + \frac{1}{2}VM^2 + 2M$$

for all $i, j \in I$. Hence, for all $n \in \mathbb{Z}_+$,

$$\|\phi_n(b_i)\| \leq (V+1)^n \left(H + \frac{1}{2}VM^2 + 2M\right)^n$$

and so $\mathcal{A}_\phi = \mathcal{A}_0$, by Corollary 2.11. Hence there is a flow on \mathcal{A} for this generator.

Example 4.12. We can lift the boundedness assumptions in Example 4.11 by taking I to be a disjoint union of subsets,

$$I = \bigsqcup_{k \in K} I_k,$$

such that there is no transport between any of these subsets, *i.e.*,

$$\alpha_{i,j} \neq 0 \text{ only if there is some } k \in K \text{ such that } i, j \in I_k.$$

Assume the symmetry condition (4.12) once again. Suppose that in each I_k the conditions of (4.14) are satisfied, but with respect to constants M_k , V_k and H_k that depend on k . Then, if $i \in I_k$, we get the estimate

$$\|\phi_n(b_i)\| \leq (V_k+1)^n \left(H_k + \frac{1}{2}V_kM_k^2 + 2M_k\right)^n$$

and so $\mathcal{A}_\phi = \mathcal{A}_0$ once more, but now it is possible that $M = \infty$ *et cetera*.

Example 4.13. To create an example where the graph associated to I has only one component, but where we do not assume $M < \infty$ as in Example 4.11, assume once again that I is decomposed into a disjoint union:

$$I = \bigsqcup_{k \in \mathbb{Z}_+} I_k \quad \text{with } |I_k| < \infty \text{ for all } k \in \mathbb{Z}_+.$$

This time assume, as well as the symmetry condition (4.12), that $\alpha_{i,j} = 0$ unless there is some $k \in \mathbb{Z}_+$ such that $i \in I_k$ and $j \in I_{k+1}$, or $j \in I_k$ and $i \in I_{k+1}$, so that there is transport only between neighbouring levels in I . Set

$$a_k = \sup\{|\alpha_{i,j}| : i \in I_k, j \in I_{k+1}\} \quad \text{for all } k \in \mathbb{Z}_+,$$

and furthermore assume that the energies are bounded, *i.e.*, $H < \infty$.

Now if $k \in \mathbb{N}$ and $i \in I_k$ then

$$\begin{aligned} \sum_{j \in \text{supp}^+(i)} \|B_{i,j}\| &\leq \|B_{i,i}\| + \sum_{j \in I_{k-1}} \|B_{i,j}\| + \sum_{j \in I_{k+1}} \|B_{i,j}\| \\ &\leq |\lambda_i| + 2|I_{k-1}|a_{k-1} + 2|I_{k+1}|a_k, \end{aligned}$$

with a similar estimate holding if $i \in I_0$. Furthermore,

$$|\lambda_i| \leq H + \frac{1}{2}|I_{k-1}|a_{k-1}^2 + \frac{1}{2}|I_{k+1}|a_k^2.$$

As in Example 4.11, if it can be shown that

$$\sum_{j \in \text{supp}^+(i)} \|B_{i,j}\| \leq C$$

for some constant C that does not depend on i , it follows that $\|\phi_n(b_i)\| \leq C^n$ for each $n \in \mathbb{Z}_+$ and $i \in I$, and so $\mathcal{A}_\phi = \mathcal{A}_0$ once more. Here, the previous working shows this will hold if there are constants $a > 0$, $b > 0$ and $p \geq 1$ such that

$$a_k \leq \frac{a}{(k+2)^p} \quad \text{and} \quad |I_k| \leq b(k+1)^p \quad \text{for all } k \in \mathbb{Z}_+.$$

It is clear that this can yield an example where $M = \infty$, *i.e.*, there is no upper bound on the valencies.

5 Flows on universal C^* algebras

5.1 The non-commutative torus

Definition 5.1. Let $\lambda \in \mathbb{T}$, the set of complex numbers with unit modulus. The *non-commutative torus* is the universal C^* algebra \mathcal{A} generated by unitaries U and V which satisfy the relation

$$UV = \lambda VU.$$

Let \mathcal{A}_0 denote the dense $*$ -subalgebra of \mathcal{A} generated by U and V .

There is a faithful trace tr on \mathcal{A} such that $\tau(U^m V^n) = \mathbb{1}_{m=n=0}$ for all $m, n \in \mathbb{Z}$; the proof of this in [7, pp.166–168] is valid for all λ . Consequently $\{U^m V^n : m, n \in \mathbb{Z}\}$ is a basis for \mathcal{A}_0 .

Lemma 5.2. Let $\mathfrak{h} := \ell^2(\mathbb{Z}^2)$, let

$$(U_c u)_{m,n} = u_{m+1,n} \quad \text{and} \quad (V_c u)_{m,n} = \lambda^m u_{m,n+1} \quad \text{for all } u \in \mathfrak{h} \text{ and } m, n \in \mathbb{Z},$$

and let $\mathcal{A}_c \subseteq \mathcal{B}(\mathfrak{h})$ be the C^* algebra generated by U_c and V_c . There is a C^* isomorphism from \mathcal{A} to \mathcal{A}_c such that $U \mapsto U_c$ and $V \mapsto V_c$. Moreover, under this map the trace tr corresponds to the vector state given by $e \in \mathfrak{h}$ such that $e_{m,n} = \mathbb{1}_{m=n=0}$ for all $m, n \in \mathbb{Z}$.

Proof. Unitarity of U_c and V_c is immediately verified, as is the identity $U_c V_c = \lambda V_c U_c$, so the universality of \mathcal{A} gives a surjective $*$ -homomorphism from \mathcal{A} to \mathcal{A}_c . Injectivity is a consequence of the final observation, that tr corresponds to the vector state given by e . \square

From now on we will identify \mathcal{A} and \mathcal{A}_c .

Definition 5.3. For each $(\mu, \nu) \in \mathbb{T}^2$, let $\pi_{\mu, \nu}$ be the automorphism of \mathcal{A} such that

$$\pi_{\mu, \nu}(U^m V^n) = \mu^m \nu^n U^m V^n \quad \text{for all } m, n \in \mathbb{Z};$$

the existence of $\pi_{\mu, \nu}$ is an immediate consequence of universality.

The proofs of the next two lemmas are a matter of routine algebraic computation.

Lemma 5.4. For all $a, b \in \mathbb{Z}$, define maps ${}_a\delta : \mathcal{A}_0 \rightarrow \mathcal{A}_0$ and $\delta_b : \mathcal{A}_0 \rightarrow \mathcal{A}_0$ by linear extension of the identities

$${}_a\delta(U^m V^n) = m U^{a+m} V^n \quad \text{and} \quad \delta_b(U^m V^n) = n \lambda^{-bm} U^m V^{b+n} \quad \text{for all } m, n \in \mathbb{Z}.$$

Then ${}_a\delta$ is a π_{1, λ^a} -derivation and δ_b is a $\pi_{\lambda^{-b}, 1}$ -derivation; moreover, their adjoints are such that

$${}_a\delta^\dagger(U^m V^n) = -m \lambda^{an} U^{-a+m} V^n \quad \text{and} \quad \delta_b^\dagger(U^m V^n) = -n U^m V^{-b+n}$$

for all $m, n \in \mathbb{Z}$.

Remark 5.5. The sufficient condition in Lemma 5.4 is also necessary. It is easy to show that if ${}_a\delta$ is a $\pi_{\mu, \nu}$ -derivation then $\mu = 1$ and $\nu = \lambda^a$; similarly, if δ_b is a $\pi_{\mu, \nu}$ -derivation then $\mu = \lambda^{-b}$ and $\nu = 1$.

Lemma 5.6. With \mathcal{A}_0 as in Definition 5.1, and ${}_a\delta$ and δ_b as in Lemma 5.4, fix $c_1, c_2 \in \mathbb{C}$ and let

$$\phi : \mathcal{A}_0 \rightarrow \mathcal{A}_0 \otimes \mathcal{B}(\mathbb{C}^3); \quad x \mapsto \begin{bmatrix} \tau(x) & \overline{c_1} {}_a\delta^\dagger(x) & \overline{c_2} \delta_b^\dagger(x) \\ c_1 {}_a\delta(x) & \pi_{1, \lambda^a}(x) - x & 0 \\ c_2 \delta_b(x) & 0 & \pi_{\lambda^{-b}, 1}(x) - x \end{bmatrix},$$

where the map

$$\tau : \mathcal{A}_0 \rightarrow \mathcal{A}_0; \quad U^m V^n \mapsto -\frac{1}{2}(|c_1|^2 m^2 + |c_2|^2 n^2) U^m V^n.$$

Then τ is $*$ -linear and ϕ is a flow generator.

Lemma 5.7. Let ϕ be as in Lemma 5.6. If $a = b = 0$ then $\mathcal{A}_\phi = \mathcal{A}_0$; conversely, if $a \neq 0$ and $c_1 \neq 0$ then $U \notin \mathcal{A}_\phi$, and if $b \neq 0$ and $c_2 \neq 0$ then $V \notin \mathcal{A}_\phi$.

Proof. When $a = b = 0$, note that $\phi(U) = U \otimes m_U$ and $\phi(V) = V \otimes m_V$, where

$$m_U := \begin{bmatrix} -\frac{1}{2}|c_1|^2 & -\overline{c_1} & 0 \\ c_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad m_V := \begin{bmatrix} -\frac{1}{2}|c_2|^2 & 0 & -\overline{c_2} \\ 0 & 0 & 0 \\ c_2 & 0 & 0 \end{bmatrix}.$$

Hence $\phi_n(U) = U \otimes m_U^{\otimes n}$ and $\phi_n(V) = V \otimes m_V^{\otimes n}$, so $U, V \in \mathcal{A}_\phi$, as claimed, and $\mathcal{A}_\phi = \mathcal{A}_0$, by Corollary 2.11.

If $a > 0$ then, by induction, one gets that

$${}_a\delta^n(U) = \prod_{i=0}^{n-1} (ia + 1) U^{an+1} \quad \text{for all } n \in \mathbb{N}.$$

Let $e = [1 \ 0 \ 0]^T$ and $f = [0 \ 1 \ 0]^T$ be unit vectors in \mathbb{C}^3 , and note that

$$(1_{\mathfrak{h}} \otimes \langle f | \otimes \cdots \otimes \langle f |) \phi_n(x) (1_{\mathfrak{h}} \otimes |e\rangle \otimes \cdots \otimes |e\rangle) = c_1^n {}_a\delta^n(x) \quad \text{for all } x \in \mathcal{A}_0,$$

so

$$\|\phi_n(U)\| \geq |c_1|^n \prod_{i=0}^{n-1} (ia + 1) \geq |c_1|^n n!.$$

If $a < 0$ then, by considering ${}_a\delta^\dagger$ instead, we see that

$$\|\phi_n(U)\| \geq \|(1_{\mathfrak{h}} \otimes \langle e | \otimes \cdots \otimes \langle e |) \phi_n(U) (1_{\mathfrak{h}} \otimes |f\rangle \otimes \cdots \otimes |f\rangle)\| \geq |c_1|^n n!.$$

A similar proof shows that $V \notin \mathcal{A}_\phi$ when $b \neq 0$. □

Remark 5.8. The lower bounds obtained in Lemma 5.7 when $a \neq 0$ or $b \neq 0$ show that our techniques do not apply in these cases. The same problem arises if one attempts to use the results of [9] instead.

The following theorem gives the existence of a quantum flow used by Goswami, Sahu and Sinha [12, Theorem 2.1(i)].

Theorem 5.9. *Let \mathcal{A} be as in Definition 5.1 and ϕ as in Lemma 5.6 for $a = b = 0$. There exists an adapted family j of unital $*$ -homomorphisms from \mathcal{A} to $\mathcal{B}(\mathfrak{h} \bar{\otimes} \mathcal{F})$ such that*

$$\langle u\varepsilon(f), j_t(x)v\varepsilon(g) \rangle = \langle u\varepsilon(f), (xv)\varepsilon(g) \rangle + \int_0^t \langle u\varepsilon(f), j_s(\phi_{\hat{g}(s)}^{\hat{f}(s)}(x))v\varepsilon(g) \rangle ds$$

for all $u, v \in \mathfrak{h}$, $f, g \in L^2(\mathbb{R}_+; \mathfrak{k})$, $x \in \mathcal{A}_0$ and $t \geq 0$.

Proof. This follows from Theorem 3.12, Lemma 5.6 and Lemma 5.7. □

Remark 5.10. The cocycle constructed in Theorem 5.9 is essentially a classical object: as noted in [5, Theorem 2.1], when $c_1 = c_2 = i$ one may take

$$j_t(x) := \beta(\exp(2\pi i B_t^1), \exp(2\pi i B_t^2))(x) \quad \text{for all } x \in \mathcal{A} \text{ and } t \geq 0,$$

where $\beta : \mathbb{T}^2 \rightarrow \text{Aut}(\mathcal{A})$ is the natural action of the 2-torus \mathbb{T}^2 on \mathcal{A} , so that

$$\beta(z, w)(U^m V^n) = z^m w^n U^m V^n \quad \text{for all } (z, w) \in \mathbb{T}^2,$$

and the Fock space \mathcal{F} is identified in the usual manner with the L^2 space of the two-dimensional classical Brownian motion (B^1, B^2) .

The existence of flows where the generator has non-zero gauge part may also be established.

Lemma 5.11. *Fix $(\mu, \nu) \in \mathbb{T}^2$ with $\mu \neq 1$. Let \mathcal{A}_0 be as in Definition 5.1 and $\pi_{\mu, \nu}$ as in Definition 5.3. There exists a flow generator*

$$\phi : \mathcal{A}_0 \rightarrow \mathcal{A}_0 \otimes \mathcal{B}(\mathbb{C}^2); \quad x \mapsto \begin{bmatrix} \tau(x) & -\mu\delta(x) \\ \delta(x) & \pi_{\mu, \nu}(x) - x \end{bmatrix},$$

where the $\pi_{\mu, \nu}$ -derivation

$$\delta : \mathcal{A}_0 \rightarrow \mathcal{A}_0; \quad U^m V^n \mapsto \frac{1 - \mu^m \nu^n}{1 - \mu} U^m V^n \quad (5.1)$$

is such that $\delta^\dagger = -\mu\delta$, and the map

$$\tau := \frac{\mu}{1 - \mu} \delta : \mathcal{A}_0 \rightarrow \mathcal{A}_0; \quad U^m V^n \mapsto \frac{\mu(1 - \mu^m \nu^n)}{(1 - \mu)^2} U^m V^n.$$

Furthermore, $U, V \in \mathcal{A}_\phi$ and so $\mathcal{A}_\phi = \mathcal{A}_0$.

Proof. Using the basis $\{U^m V^n : m, n \in \mathbb{Z}\}$, one can readily verify that δ is a $\pi_{\mu, \nu}$ -derivation such that $\delta^\dagger = -\mu\delta$, and hence ϕ is a flow generator. Since

$$\phi(U) = U \otimes \begin{bmatrix} \frac{\mu}{1 - \mu} & -\mu \\ 1 & \mu - 1 \end{bmatrix} \quad \text{and} \quad \phi(V) = V \otimes \frac{1 - \nu}{1 - \mu} \begin{bmatrix} \frac{\mu}{1 - \mu} & -\mu \\ 1 & \mu - 1 \end{bmatrix},$$

the fact that $\{U, V\} \subseteq \mathcal{A}_\phi$ follows as in the proof of Lemma 5.7. \square

Remark 5.12. It is curious to note that for ϕ as in Lemma 5.11 we have $\tau = \mu(1 - \mu)^{-1}\delta$, and so τ is first rather than second order. Whether or not ϕ or, equivalently, δ is bounded is an open problem; our existence result obviates the need to determine this.

Theorem 5.13. *Let \mathcal{A} be as in Definition 5.1 and ϕ as in Lemma 5.11. There exists an adapted family j of unital $*$ -homomorphisms from \mathcal{A} to $\mathcal{B}(\mathfrak{h} \bar{\otimes} \mathcal{F})$ such that*

$$\langle u\varepsilon(f), j_t(x)v\varepsilon(g) \rangle = \langle u\varepsilon(f), (xv)\varepsilon(g) \rangle + \int_0^t \langle u\varepsilon(f), j_s(\phi_{\hat{g}(s)}^{\hat{f}(s)}(x))v\varepsilon(g) \rangle ds$$

for all $u, v \in \mathfrak{h}$, $f, g \in L^2(\mathbb{R}_+; \mathfrak{k})$, $x \in \mathcal{A}_0$ and $t \geq 0$.

As noted by Hudson and Robinson [14], the following result makes clear why in Theorem 5.9 it is necessary to consider separately the derivations $c_1 \delta$ and $c_2 \delta_0$, rather than considering the sum $\delta = c_1 \delta + c_2 \delta_0$.

Proposition 5.14. *Let δ and δ_0 be as in Lemma 5.4, and let $\delta = c_1 \delta + c_2 \delta_0$ for complex numbers c_1 and c_2 . A necessary and sufficient condition for the existence of a linear map $\tau : \mathcal{A}_0 \rightarrow \mathcal{A}$ such that*

$$\tau(xy) - \tau(x)y - x\tau(y) = \delta^\dagger(x)\delta(y) \quad \text{for all } x, y \in \mathcal{A}_0$$

is the equality $c_1 \overline{c_2} = \overline{c_1} c_2$.

Proof. This may be established by adapting slightly the proof of [22, Theorem 2.2]. \square

5.2 The universal rotation algebra

To avoid the issue of Proposition 5.14, Hudson and Robinson work with the universal rotation algebra.

Definition 5.15. Let \mathcal{A} be the *universal rotation algebra* [2]: this is the universal C^* algebra with unitary generators U , V and Z satisfying the relations

$$UV = ZVU, \quad UZ = ZU \quad \text{and} \quad VZ = ZV.$$

It may be viewed as the group C^* algebra corresponding to the discrete Heisenberg group

$$\Gamma := \langle u, v, z \mid uv = zvu, \quad uz = zu, \quad vz = zv \rangle;$$

from this perspective, its universal nature is immediately apparent.

Letting \mathcal{A}_0 denote the $*$ -subalgebra generated by U , V and Z , there are skew-adjoint derivations

$$\delta_1 : \mathcal{A}_0 \rightarrow \mathcal{A}_0; \quad U^m V^n Z^p \mapsto mU^m V^n Z^p \quad \text{and} \quad \delta_2 : \mathcal{A}_0 \rightarrow \mathcal{A}_0; \quad U^m V^n Z^p \mapsto nU^m V^n Z^p$$

for all $m, n, p \in \mathbb{Z}$.

Remark 5.16. For a concrete version of the universal rotation algebra, let $\mathfrak{h} := \ell^2(\mathbb{Z}^3)$ and define operators U_c , V_c and Z_c by setting

$$(U_c u)_{m,n,p} = u_{m+1,n,p}, \quad (V_c u)_{m,n,p} = u_{m,n+1,m+p} \quad \text{and} \quad (Z_c u)_{m,n,p} = u_{m,n,p+1}$$

for all $u \in \mathfrak{h}$ and $m, n, p \in \mathbb{Z}$. It is readily verified that U_c , V_c and Z_c are unitary and satisfy the commutation relations as claimed; let \mathcal{A}_c be the C^* algebra generated by these operators.

Universality gives a surjective $*$ -homomorphism from \mathcal{A} to \mathcal{A}_c such that $U \mapsto U_c$, $V \mapsto V_c$ and $Z \mapsto Z_c$, and injectivity may be established in the same manner as for the non-commutative torus: there is a faithful state τ on \mathcal{A} such that $\tau(U^m V^n Z^p) = \mathbb{1}_{m=n=p=0}$ and this corresponds to the vector state given by $e \in \mathfrak{h}$ such that $e_{m,n,p} = \mathbb{1}_{m=n=p=0}$.

Lemma 5.17. With \mathcal{A}_0 , δ_1 and δ_2 as in Definition 5.15, fix $c_1, c_2 \in \mathbb{C}$, let $\delta = c_1 \delta_1 + c_2 \delta_2$ and define the Bellissard map

$$\tau : \mathcal{A}_0 \rightarrow \mathcal{A}_0; \quad U^m V^n Z^p \mapsto -\left(\frac{1}{2}|c_1|^2 m^2 + \frac{1}{2}|c_2|^2 n^2 + \overline{c_1} c_2 mn + (\overline{c_1} c_2 - c_1 \overline{c_2})p\right) U^m V^n Z^p,$$

Then τ is $*$ -linear and such that

$$\tau(xy) - \tau(x)y - x\tau(y) = \delta^\dagger(x)\delta(y) \quad \text{for all } x, y \in \mathcal{A}_0,$$

so the map

$$\phi : \mathcal{A}_0 \rightarrow \mathcal{A}_0 \otimes \mathcal{B}(\mathbb{C}^2); \quad x \mapsto \begin{bmatrix} \tau(x) & \delta^\dagger(x) \\ \delta(x) & 0 \end{bmatrix}$$

is a flow generator.

Furthermore, $U, V, Z \in \mathcal{A}_\phi$ and $\mathcal{A}_\phi = \mathcal{A}_0$.

Proof. The algebraic statements are readily verified, and a short calculation shows that

$$\phi(U) = U \otimes m_U, \quad \phi(V) = V \otimes m_V \quad \text{and} \quad \phi(Z) = Z \otimes m_Z,$$

where

$$m_U = \begin{bmatrix} -\frac{1}{2}|c_1|^2 & -\overline{c_1} \\ c_1 & 0 \end{bmatrix}, \quad m_V = \begin{bmatrix} -\frac{1}{2}|c_2|^2 & -\overline{c_2} \\ c_2 & 0 \end{bmatrix} \quad \text{and} \quad m_Z = \begin{bmatrix} c_1\overline{c_2} - \overline{c_1}c_2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence

$$\phi_n(U) = U \otimes m_U^{\otimes n}, \quad \phi_n(V) = V \otimes m_V^{\otimes n} \quad \text{and} \quad \phi_n(Z) = Z \otimes m_Z^{\otimes n}$$

for all $n \in \mathbb{Z}_+$, so $U, V, Z \in \mathcal{A}_\phi$ and $\mathcal{A}_\phi = \mathcal{A}_0$, by Corollary 2.11. \square

The following theorem is an algebraic version of the result presented by Hudson and Robinson in [14, Section 4].

Theorem 5.18. *Let \mathcal{A} be as in Definition 5.15 and ϕ as in Lemma 5.17. There exists an adapted family j of unital $*$ -homomorphisms from \mathcal{A} to $\mathcal{B}(\mathfrak{h} \bar{\otimes} \mathcal{F})$ such that*

$$\langle u\varepsilon(f), j_t(x)v\varepsilon(g) \rangle = \langle u\varepsilon(f), (xv)\varepsilon(g) \rangle + \int_0^t \langle u\varepsilon(f), j_s(\phi_{\hat{g}(s)}^{\hat{f}(s)}(x))v\varepsilon(g) \rangle ds$$

for all $u, v \in \mathfrak{h}$, $f, g \in L^2(\mathbb{R}_+; \mathfrak{k})$, $x \in \mathcal{A}_0$ and $t \geq 0$. \square

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6 References

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